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An Optimal Mesh Generator for Lagrangian Hydrodynamic Calculations in Two Space Dimensions*

W. D. BARFIELD

University of California, Los Alamos Scientific Laboratory, Los Alamos, New Mexico 87544

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A linear numerical method is described which generates a curvilinear mesh that is as nearly orthogonal as possible, in a least-squares sense, subject to the constraint that mesh lines are matched to arbitrarily specified points around the perimeter of the region. An option permits trading off more orthogonality for improved line spacing. The use of the method in a practical zoner/rezoner is discussed. The method also has application for Eulerian hydrodynamic calculations.

INTRODUCTION

The difference equations of Lagrangian hydrodynamics in two space dimensions are conveniently referred to an orthogonal rectilinear mesh of equally spaced lines (e.g., Fig. 1), commonly called the "logical mesh." Let the lines be labeled with integer values of two variables K, L. The image in Eulerian space of a sub-

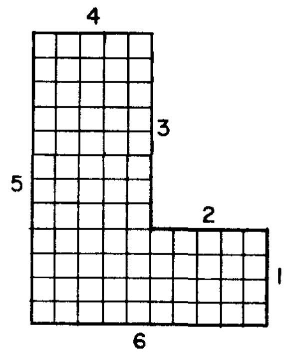

FIG. 1. L-shaped portion of a logical mesh.

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region[1] of the logical mesh is a curvilinear mesh inscribed in the corresponding subregion of Eulerian (R, Z) space. The "K lines" and "L lines" move with the fluid: No fluid crosses the mesh.

The problem of generating a mesh is equivalent to finding a mapping of the subregion of the logical mesh onto the corresponding region of R–Z space. Characteristics of an ideal mesh can be enumerated:

1. The K lines and/or L lines should not be too crowded or spaced too far apart: In the one case the time interval Δt for the calculation will be unnecessarily restricted; in the other the spatial resolution will suffer.

2. The K lines should not be too skew with respect to the L lines: It can be argued that the accuracy of finite-difference approximations to the differential equations deteriorates as the mesh becomes increasingly nonorthogonal.

3. The mesh should be smooth, with first derivatives everywhere continuous.

4. The mesh lines should be orthogonal to the boundary of the subregion, assuring a smooth join to lines in adjacent subregions.

5. The mapping should enable mesh lines to be matched around the perimeter of the subregion to lines in adjacent subregions. (That is, it should be possible to match the lines to arbitrarily specified points around the perimeter.)

6. The method should enable inscribing a mesh in subregions of arbitrary shapes (bounded by portions of K and L lines).

Obviously, many calculations have been done utilizing meshes that are not ideal in the above sense.

A method for inscribing an orthogonal curvilinear mesh (OCM) in an arbitrary subregion was described in a previous article [1]. The method has the limitation that mesh lines cannot be matched around the entire perimeter of a subregion. For example, in the L-shaped subregion of a logical mesh shown in Fig. 1, lines could be matched on sides 1, 2, 3, 4, but not sides 5 and 6. (The "L" could be subdivided into three rectangles, enabling matching along sides 5 and 6 instead of 1, 2, 3, 4.)

The present article presents a method for inscribing a mesh in an arbitrary subregion which relaxes the requirement of strict orthogonality to permit matching lines to arbitrarily selected points along all sides of the subregion. The method to be described also permits trading off more orthogonality for improved spacing of lines.

[1] A single material or portion of a single material.

AN OPTIMAL MESH GENERATOR

A measure of the deviation of a mesh from orthogonality is given by a positive integral

$$I_1(\alpha) = \iint [(R_K - \alpha Z_L)^2 + (Z_K + \alpha R_L)^2]\, dK\, dL.$$

The subscripts K, L indicate partial derivatives, and the integral is over the subregion of interest. In the case of a strictly orthogonal mesh the integral vanishes, since at each point

$$(R + iZ)_K = -i\alpha(R + iZ)_L, \tag{1}$$

i.e., the vector $(R + iZ)_K$ tangent to a line $L = \text{const}$ is perpendicular to $(R + iZ)_L$, tangent to $K = \text{const}$ at the same point. The parameter α represents the aspect ratio (length/width) of the mesh quadrilaterals.[2]

We wish to minimize $I_1(\alpha)$ subject to the boundary conditions, the given spacing of points around the perimeter. If $\alpha = \text{const}$ is assumed, the problem is linear. The Euler equations minimizing I_1 are

$$\begin{cases} R_{KK} + \alpha^2 R_{LL} = 0, \\ Z_{KK} + \alpha^2 Z_{LL} = 0. \end{cases}$$

These Laplace-like equations are readily solvable by standard numerical techniques for elliptic equations. In this case α represents an average aspect ratio.

Another positive integral, which essentially measures the deviation of the mesh quadrilaterals from parallelograms, has been proposed by Dr. G. Fraley:

$$I_2 = \iint |(R + iZ)_{KL}|^2\, dK\, dL.$$

Minimizing I_2 tends to make adjacent K lines or L lines "parallel," i.e., equidistant along their lengths.

The present method minimizes a weighted sum of the above integrals,

$$w_1 I_1(\alpha) + w_2 I_2\,.$$

With constant α and $w_2 \neq 0$ the resulting Euler equations are fourth-order,

$$\begin{cases} w_2 R_{KKLL} + w_1(R_{KK} + \alpha^2 R_{LL}) = 0, \\ w_2 Z_{KKLL} + w_1(Z_{KK} + \alpha^2 Z_{LL}) = 0. \end{cases} \tag{2}$$

The weights and the parameter α are arbitrary. The resulting meshes are optimal in a least-squares sense.

[2] For a strictly orthogonal mesh, $\alpha^2 = (R_K{}^2 + Z_K{}^2)/(R_L{}^2 + Z_L{}^2)$. The image in R–Z space of a mesh unit bounded by K, $K + 1$, L, $L + 1$ is imprecisely referred to as a "quadrilaterial." The mesh lines in R–Z space are curves, of course.

Equations (2) are hyperbolic. The K lines and L lines are characteristic: Discontinuities in the derivatives at points of the boundary "propagate" along the corresponding K or L lines, as can be seen from the examples given below.[3]

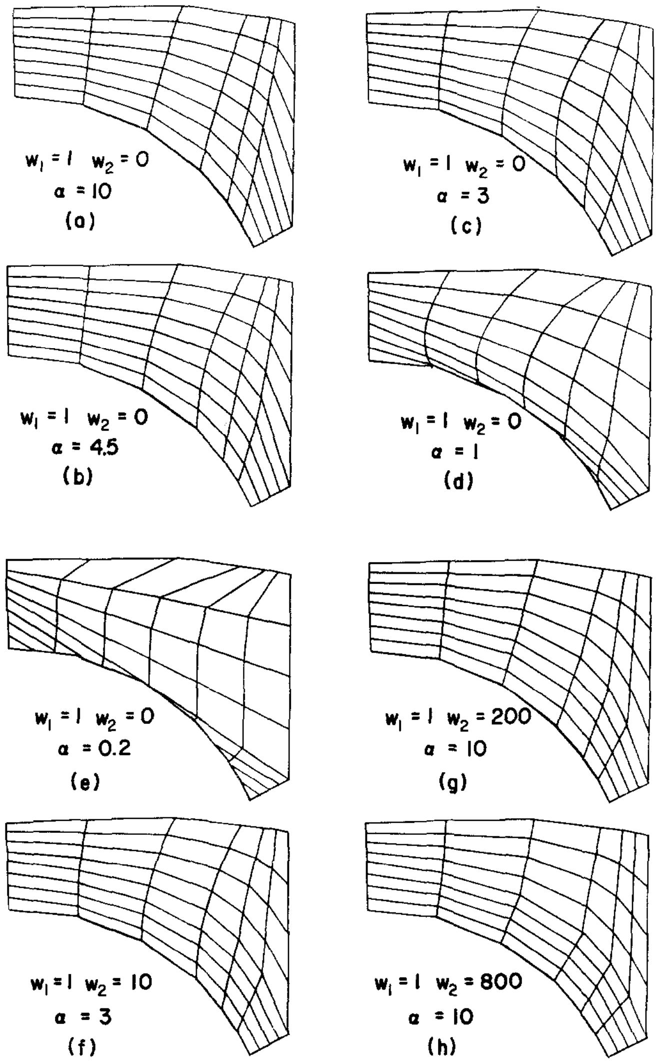


FIGURE 2

[3] It was pointed out to the author by Dr. G. Pimbley that Eqs. (2) with $w_1 = 0$ can be readily integrated analytically when the subregion of K–L space is a rectangle. See also Ref. [2]. The analytical solution was used to check the numerical solutions included in this article.

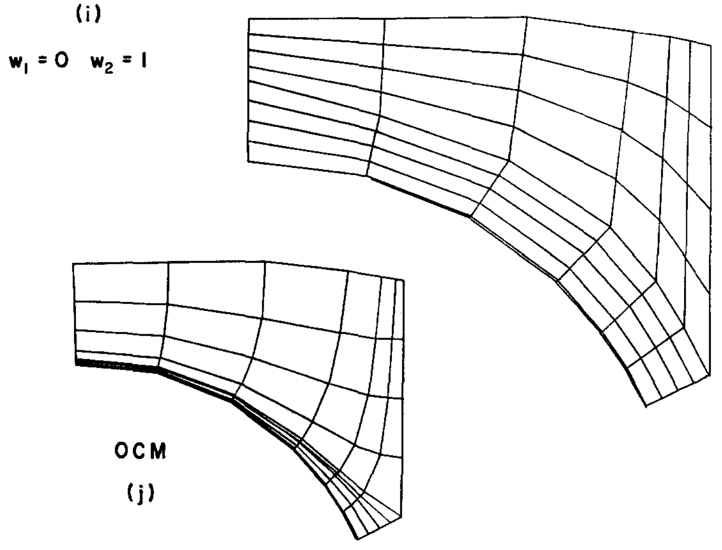


FIG. 2. $\alpha^* \approx 3$.

Of course, it is not necessary to assume α = constant. In fact, it seems reasonable to suppose that it might be possible to determine an α field, $\alpha(K, L)$, so as to make the orthogonality integral I_1 *vanish.* Numerical solutions of the corresponding quasilinear mapping problem indicate that this is possible, at least in some cases.[4]

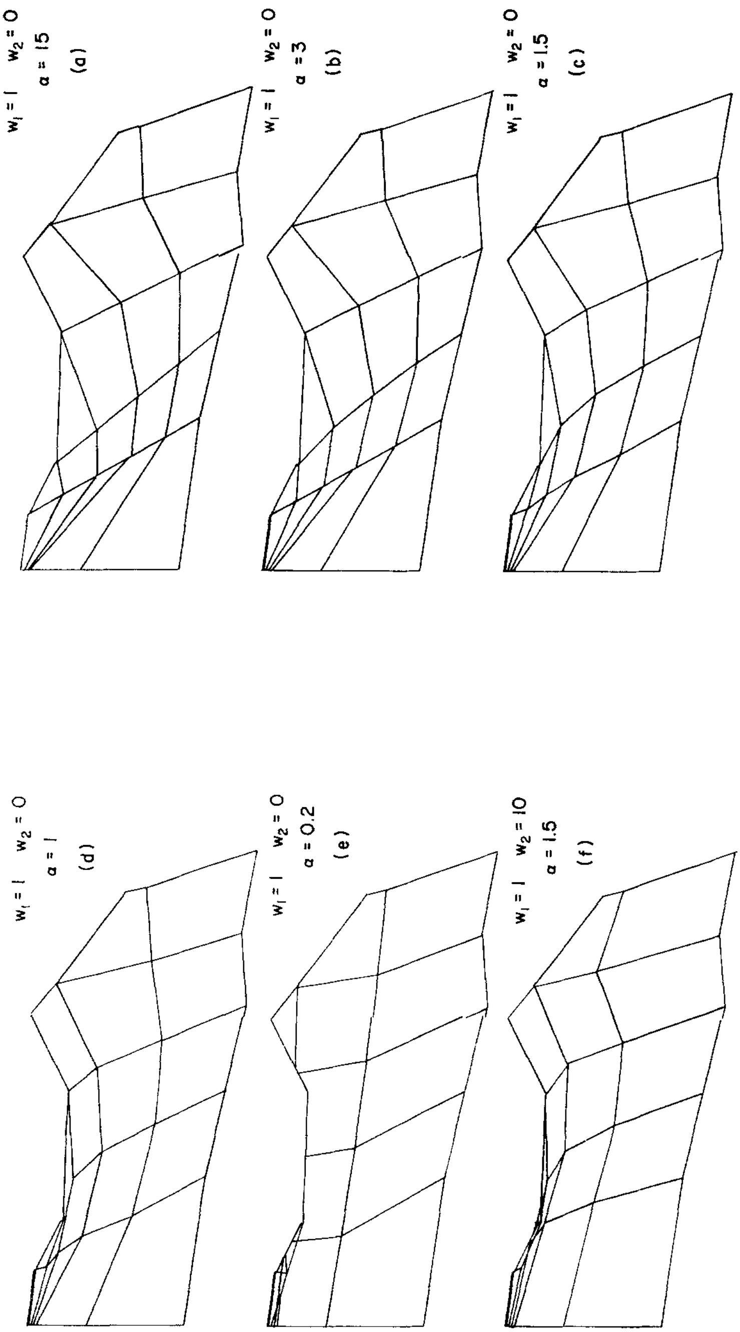
EXAMPLES

The Eqs. (2) were solved numerically by the method of successive over-relaxation. A series of runs varying the value of the S.O.R. parameter ω indicated that the curve of number of iterations to converge vs ω usually bottoms out at a value of ω not too different from the best value for Laplace's equation solved on a rectangle circumscribing the subregion of K–L space. The number of iterations required for convergence is not a terribly sensitive function of ω. In $w_1 = 0$ cases convergence is relatively slow (a few hundred iterations required for typical cases).

Figures 2–8[5] are an assortment of examples, including some unsatisfactory zonings, illustrating the effects of various choices of the parameter α and the weights w_1, w_2. Also shown for comparison are examples of OCM (generated

[4] If the method turns out to be generally useful, it will be the subject of a future article.

[5] The figures were produced by an automatic plotting machine which approximates the curved mesh lines by straight line segments.



$w_1 = 1$ $w_2 = 0$
$\alpha = 15$
(a)
$w_1 = 1$ $w_2 = 0$
$\alpha = 3$
(b)
$w_1 = 1$ $w_2 = 0$
$\alpha = 1.5$
(c)
$w_1 = 1$ $w_2 = 0$
$\alpha = 1$
(d)
$w_1 = 1$ $w_2 = 0$
$\alpha = 0.2$
(e)
$w_1 = 1$ $w_2 = 10$
$\alpha = 1.5$
(f)

FIGURE 3

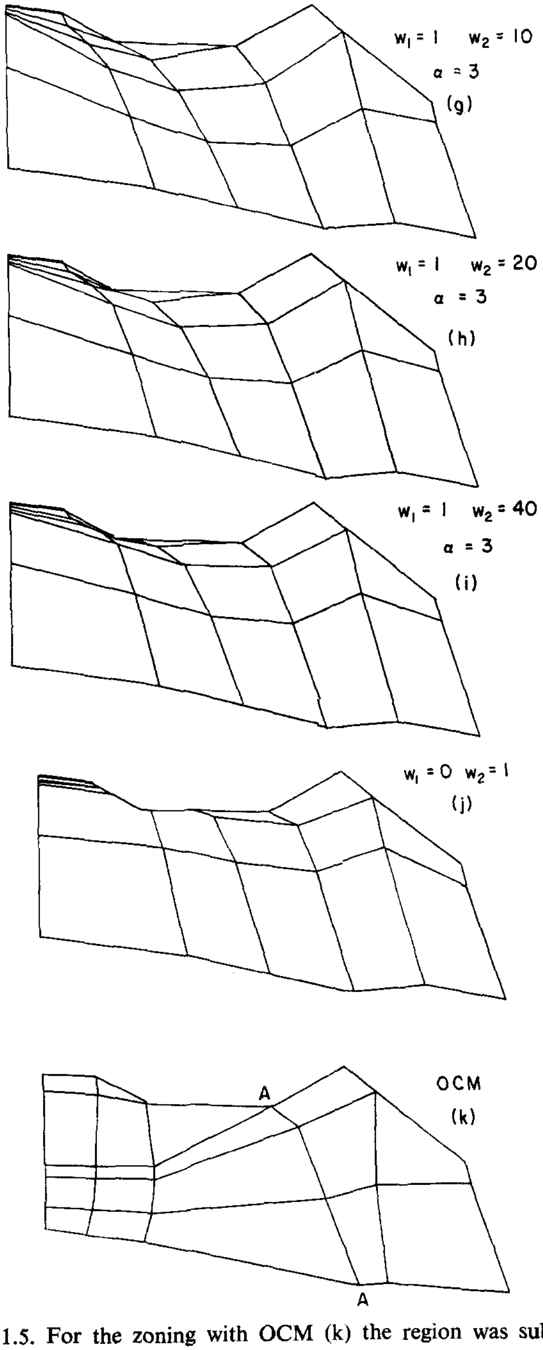


FIG. 3. $\alpha^* \approx 1.5$. For the zoning with OCM (k) the region was subdivided along A–A.

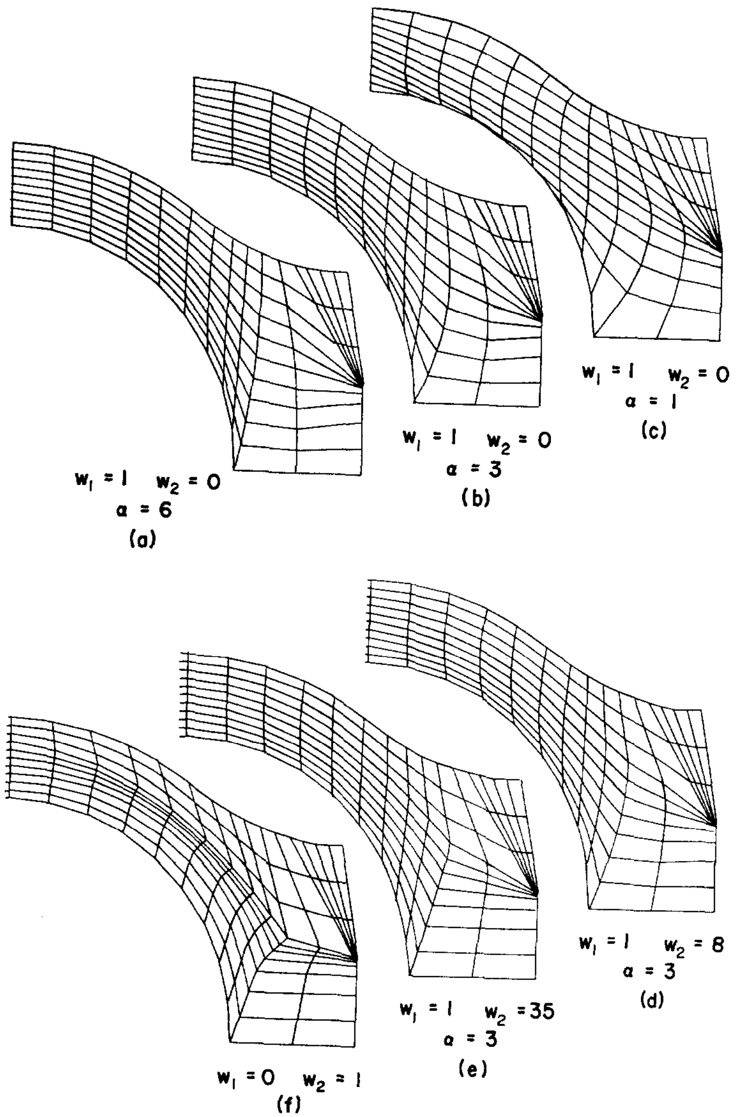


FIG. 4. $\alpha^* \approx 2.5$.

by the method of Ref. [1]) inscribed in the same subregions. The spacings of the mesh lines along the perimeters were regarded as given data. (In the examples of OCM, the line spacings were prescribed only along parts of the perimeters, as discussed above.)

It is seen that a too small (large) value of α results in a "spill over" across an inward-bulging portion of the boundary that is an L line (K line). For example, for sufficiently small α the L lines become straight lines connecting boundary points on opposite sides of the subregion; in this case, some of the L lines will

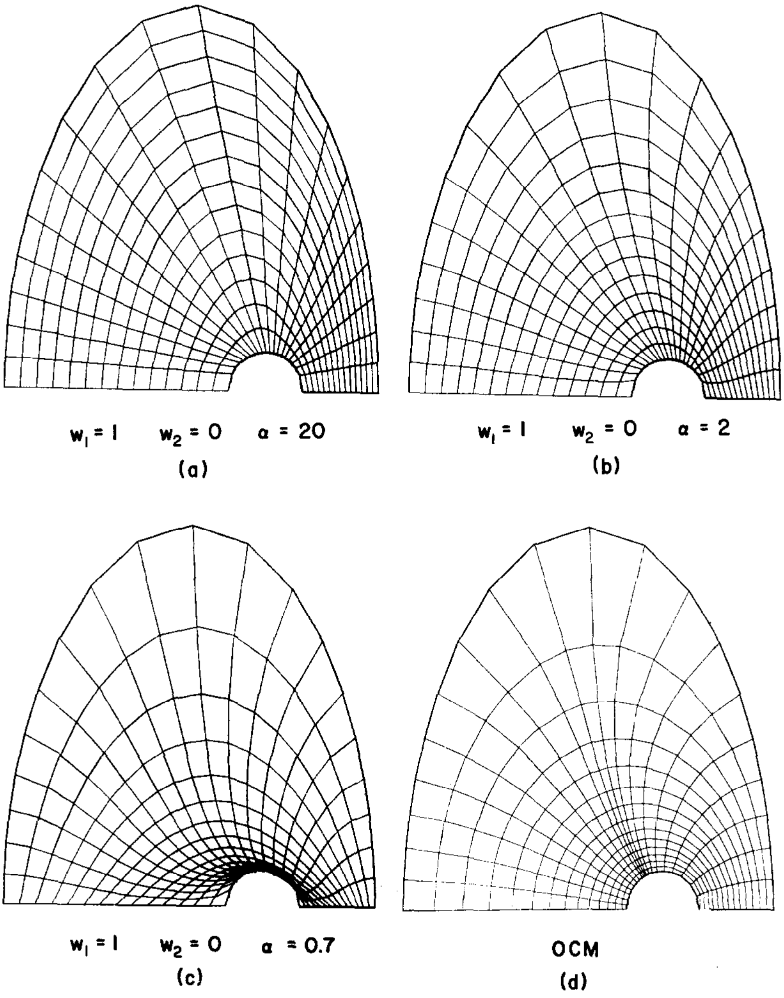


FIG. 5. $\alpha^* \approx 2$.

probably intersect an inward-bulging portion of the boundary corresponding to an L line. The value designated α^* is an estimate of

$$\frac{\langle R_K^2 + Z_K^2 \rangle^{1/2}}{\langle R_L^2 + Z_L^2 \rangle^{1/2}},$$

the indicated averages being taken over the subregion.

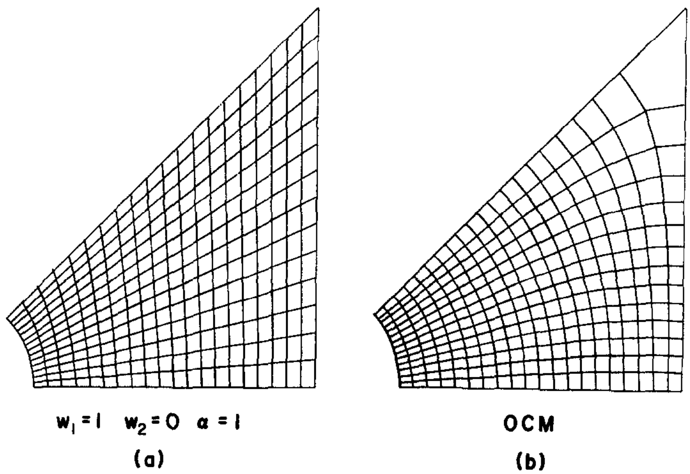


FIG. 6. $\alpha^* \approx 1$.

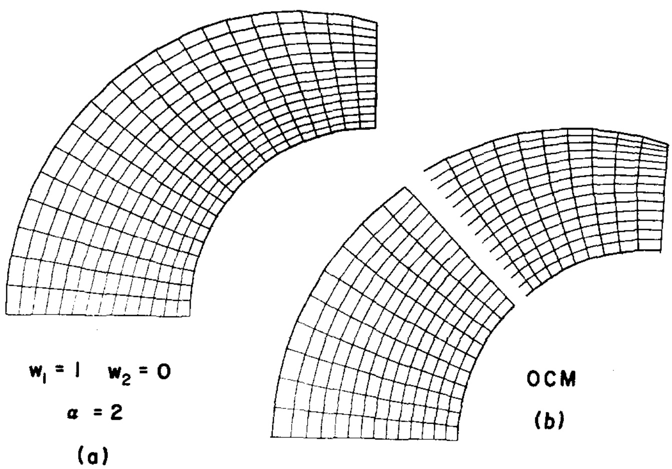


FIG. 7. $\alpha^* \approx 2$.

A Practical Zoner/Rezoner

Since orthogonal meshes have important advantages when they can be used, the given region might first be zoned with an OCM. As described in Ref. [1], it may be necessary to subdivide the region in order to accomplish this. Portions of the OCM considered to have unsuitable line spacing can be rezoned by the method presented above. Alternatively, the entire given region can be zoned by the method of this article.

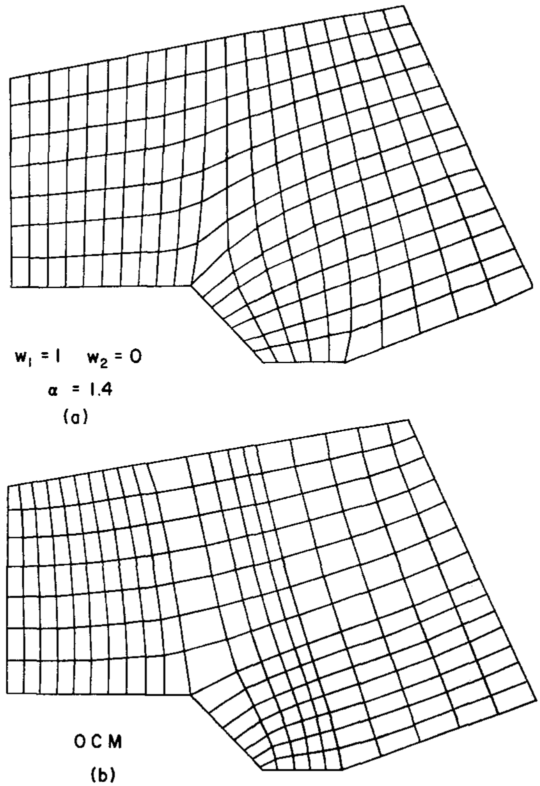


Fig. 8. $\alpha^* \approx 1$.

The parameter α and the weights w_1, w_2 give the method a flexibility and allow the user a degree of control over the average aspect ratio of the mesh cells and the line spacing not possible with a parameter-free zoner, such as that of Ref. [3]. It does not seem possible in general to determine a priori best values of the parameters. For certain simple geometric figures (Fig. 9) the best value of α is given by

α^*. By consideration of an average K line and an average L line, an estimate of α^* can be readily made for a given region. Optimal values of w_1, w_2 can be determined by inspection from a series of meshes such as

(a) $w_1 = 1, \quad w_2 = 0, \quad \alpha = \alpha^*;$

(b) $w_1 = 1, \quad w_2 = \begin{cases} 4 & (\alpha^* \leqslant 1), \\ 4(\alpha^*)^2 & (1 < \alpha^*); \end{cases}$

(c) $w_1 = 0, \quad w_2 = 1.$

Alternatively, an automatic trial-and-error procedure can be developed to enable the computer to decide which values of the weights w_1, w_2 to use. In case a mesh is generated which spills over the boundary of a subregion, at least two "bowtied" zones are produced (Fig. 10). The computer can be programmed to generate a

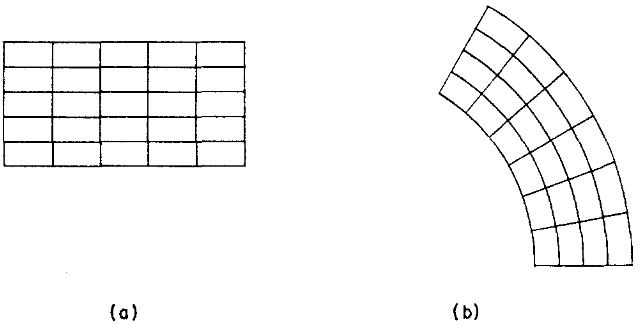


FIG. 9. Examples of regions with simple shapes for which the optimal value of the parameter α is α^*.

sequence of meshes (e.g., the sequence a–b–c given above), testing each mesh for bowties along the perimeter and any other criteria (line spacing, skewness, smoothness) deemed important. (For example, the value of the integral I_1, normalized by 2α × area of subregion in R–Z space, could be computed as a measure of skewness. For a nearly orthogonal mesh, $I_1 \leqslant 2\alpha A$.)

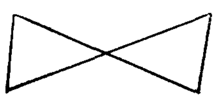


FIG. 10. "Bowtied" zone.

The present method also has application for rezoning subregions where the mesh has become too distorted as a result of the development of the flow. In such an application an estimate of α^* can be determined numerically making use of the old mesh values. OCM should be well-suited for rezoning calculations in which *slip* is allowed along material interfaces, since there would be no requirement to match lines along such interfaces.

The meshes generated by the present method also have application for Eulerian hydrodynamic calculations in which it is necessary to prescribe boundary conditions along the sides of regions with nonsimple shapes, for example, channels or rigid obstacles.

References

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3. A. M. Winslow, *J. Comput. Phys.* **2** (1967), 149.